ON THE SPECTRUM OF THE DIRICHLET LAPLACIAN IN A NARROW STRIP,

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1. Introduction

There are several reasons why the study of the spectrum of the Laplacian in a narrow neighborhood of an embedded graph is interesting. The graph can be embedded into a Euclidean space or it can be embedded into a manifold. In his pioneering work [3], Colin de Verdière used Riemannian metrics concentrated in a small neighborhood of a graph to prove that for every manifold M of dimension greater than two and for every positive number N there exists a Riemannian metric g such that the multiplicity of the smallest positive eigenvalue of the Laplacian on (M, g) equals N. Recent interest to the problem is, in particular, motivated by possible applications to mesoscopic systems. Rubinstein and Schatzman studied in [10] eigenvalues of the Neumann Laplacian in a narrow strip surrounding an embedded planar graph. The strip has constant width ϵ everywhere except neighborhoods of vertices. Under some assumptions on the structure of the strip near vertices, they proved that eigenvalues of the Neumann Laplacian converge to eigenvalues of the Laplacian on the graph. Kuchment and Zheng extended in [6] these results to the case when the strip width is not constant.

The Dirichlet boundary condition turns out to be more complicated than the Neumann condition. Eigenvalues of a domain of width ϵ are bounded from below by π^2/ϵ^2. Post studied in [9] eigenvalues $\lambda_j(\epsilon)$ of the Dirichlet Laplacian in a neighborhood of a planar graph that has constant width ϵ near the edges and that narrows down toward the vertices. He proved that $\lambda_j(\epsilon) - \pi^2/\epsilon^2$ converge to the eigenvalues of the direct sum of certain Schrödinger operators on the edges with the Dirichlet boundary conditions. We show that this result can not be extended to neighborhoods of variable width. If the width is not constant then the spectrum of the Dirichlet Laplacian is basically determined by the points where it is the widest. In the paper, we treat a simple model case: the graph is a straight segment, and the strip is the widest in one cross-section. In this case, we derive a two-term asymptotics for $\lambda_j(\epsilon)$.

We will formulate now main results of the paper. Let $h(x) > 0$ be a continuous function defined on a segment $I = [-a, b]$, where $a, b > 0$. We assume that

(i) $x = 0$ is the only point of global maximum of $h(x)$ on I;

(*ii*) The function $h(x)$ is C^1 on $I \setminus \{0\}$, and in a neighborhood of $x = 0$ it admits an expansion

$$h(x) = \begin{cases} M - c_+ x^m + O\big(x^{m+1}\big), & x > 0, \\ M - c_- |x|^m + O\big(|x|^{m+1}\big), & x < 0 \end{cases} \tag{1.1}$$

where $M, m, c_\pm$ are real numbers and $M, c_\pm > 0,\ m \geq 1$.

If $h(x)$ is C^∞ on the whole of I, then necessarily m is even and $c_- = c_+$. Another interesting case is $m = 1$ (profile of a 'broken line').

For a positive ϵ, let

$$\Omega_\epsilon = \{(x, y) : x \in I,\ 0 < y < \epsilon h(x)\}.$$

Below Δ_ϵ stands for the (positive) Dirichlet Laplacian in Ω_ϵ and $l_j(\epsilon)$ for its eigenvalues. Our main goal in this paper is to find the asymptotics of $l_j(\epsilon)$ as $\epsilon \to 0$.

In theorems 1.1 – 1.3 below the conditions (*i*) and (*ii*) are supposed to be satisfied.

Theorem 1.1. *Let $\alpha = 2(m+2)^{-1}$. Then the limits*

$$\mu_j = \lim_{\epsilon \to 0} \epsilon^{2\alpha} \left(l_j(\epsilon) - \frac{\pi^2}{M^2 \epsilon^2} \right) \tag{1.2}$$

exist, and μ_j are eigenvalues of the operator on $L^2(\mathbb{R})$ given by

$$\mathbf{H} = -\frac{d^2}{dx^2} + q(x), \qquad q(x) = \begin{cases} 2\pi^2 M^{-3} c_+ x^m, & x > 0, \\ 2\pi^2 M^{-3} c_- |x|^m, & x < 0. \end{cases} \tag{1.3}$$

Note that if $m = 2$ and $c_+ = c_-$, the operator $\mathbf{H}$ turns into the harmonic oscillator.

Our second goal is to show that the eigenvalue convergence, described by (1.2), can be obtained as a consequence of a sort of uniform convergence (i.e., convergence in norm) of the family of operators $\left(\Delta_\epsilon - \frac{\pi^2}{M^2\epsilon^2}\right)^{-1}$. The usual notion of uniform convergence does not make sense here, since for different values of ϵ the operators act in different spaces; one needs to interpret it in an appropriate way.

In $L^2(\Omega_\epsilon)$ consider the subspace $\mathfrak{L}_\epsilon$ that consists of functions

$$\psi(x, y) = \psi_\chi(x, y) = \chi(x) \sqrt{\frac{2}{\epsilon h(x)}} \sin \frac{\pi y}{\epsilon h(x)};$$

then

$$\|\psi_\chi\|^2_{L^2(\Omega_\epsilon)} = \int_I \chi^2(x) dx.$$

The mapping $\chi \mapsto \psi_\chi$ is an isometric isomorphism between $\mathfrak{L}_\epsilon$ and $L^2(I)$. With some abuse of notations, we will identify operators acting in $\mathfrak{L}_\epsilon$ with operators acting in $L^2(I)$. Obviously, $\psi_\chi \in H^{1,0}(\Omega_\epsilon)$ if $\chi \in H^{1,0}(I)$. A direct (though, rather lengthy) computation shows that

$$\int_{\Omega_\epsilon} |\nabla\psi_\chi|^2 dxdy = \int_I \chi'(x)^2 dx + \int_I \left(\frac{\pi^2}{\epsilon^2 h^2(x)} + v(x)\right)\chi^2(x)dx,$$

where

$$v(x) = \left(\frac{\pi^2}{3} + \frac{1}{4}\right)\frac{h'(x)^2}{h^2(x)}.$$

Subtracting from $\int_{\Omega_\epsilon} |\nabla\psi_\chi|^2 dxdy$ the lower bound of the resulting potential, we obtain the quadratic form (defined on $H^{1,0}(I)$)

$$\mathbf{q}_\epsilon[\chi] := \int_I \left(\chi'(x)^2 + W_\epsilon(x)\chi^2(x)\right) dx, \tag{1.4}$$

where

$$W_\epsilon(x) = \frac{\pi^2}{\epsilon^2}\left(\frac{1}{h^2(x)} - \frac{1}{M^2}\right) + v(x). \tag{1.5}$$

Since the potential $W_\epsilon(x)$ is non-negative, and it is positive for non-zero values of x, the quadratic form (1.4) is positive definite in $L^2(I)$. The self-adjoint operator on $L^2(I)$, associated with $\mathbf{q}_\epsilon$, is given by

$$\mathbf{Q}_\epsilon u = -\frac{d^2u}{dx^2} + W_\epsilon(x)u, \qquad u(-a) = u(b) = 0. \tag{1.6}$$

The result of theorem 1.2 below can be interpreted as two-term asymptotics, in a certain sense, of the operator-valued function Δ_ϵ as $\epsilon \to 0$. In its formulation, $\mathbf{I}_\epsilon$ stands for the identity operator on $L^2(\Omega_\epsilon)$.

Theorem 1.2. *There exist numbers $R_0 > 0$ and $\epsilon_0 > 0$, depending on the function h and such that*

$$\left\| \left(\Delta_\epsilon - \frac{\pi^2}{M^2\epsilon^2}\mathbf{I}_\epsilon\right)^{-1} - \mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0} \right\| \le R_0\epsilon^{3\alpha}, \qquad \forall \epsilon \in (0, \epsilon_0). \tag{1.7}$$

Here $\mathbf{0}$ is the zero operator on the subspace $\mathfrak{L}_\epsilon^\perp \subset L^2(\Omega_\epsilon)$.

The next statement describes, in what sense the operators $\mathbf{Q}_\epsilon$ approximate the operator $\mathbf{H}$ given by (1.3). Introduce the family of segments

$$I_\epsilon = (-a\epsilon^{-\alpha}, b\epsilon^{-\alpha}), \qquad \epsilon > 0$$

and define the isometry operator $\mathbf{J}_\epsilon : L^2(I) \to L^2(I_\epsilon)$ generated by the dilation $x = t\epsilon^\alpha$. We identify $L^2(I_\epsilon)$ with the subspace

$$\{u \in L^2(\mathbb{R}) : u(x) = 0 \text{ a.e. on } \mathbb{R} \setminus I_\epsilon\}.$$

If $\mathbf{Q}_\epsilon$ is the operator (1.6) in $L^2(I)$, then

$$\widehat{\mathbf{Q}}_\epsilon := \epsilon^{2\alpha} \mathbf{J}_\epsilon \mathbf{Q}_\epsilon \mathbf{J}_\epsilon^{-1} \tag{1.8}$$

is a self-adjoint operator acting in $L^2(I_\epsilon)$.

Theorem 1.3. *One has*

$$\left\| \widehat{\mathbf{Q}}_\epsilon^{-1} \oplus \mathbf{0} - \mathbf{H}^{-1} \right\| \to 0, \qquad \epsilon \to 0. \tag{1.9}$$

were $\mathbf{0}$ *is the zero operator on the subspace* $L^2(\mathbb{R} \setminus I_\epsilon)$*.*

Let us present another formulation of the latter result. It suggests an interpretation that seems to be more transparent. However, the formulation as in theorem 1.3 is more convenient for the proof.

Along with the operator $\mathbf{H}$ defined in (1.3), let us consider the operator family

$$\mathbf{H}_\epsilon = -\frac{d^2}{dx^2} + \epsilon^{-2} q(x), \quad \epsilon > 0$$

on $L^2(\mathbb{R})$, so that in particular $\mathbf{H}_1 = \mathbf{H}$. The substitution $x = t\epsilon^\alpha$ shows that $\epsilon^{2\alpha}\mathbf{H}_\epsilon$ is an isospectral family of operators. The result of theorem 1.3 can be rewritten as

$$\|(\epsilon^{2\alpha}\mathbf{Q}_\epsilon)^{-1} \oplus 0 - (\epsilon^{2\alpha}\mathbf{H}_\epsilon)^{-1}\| \to 0, \qquad \epsilon \to 0.$$

This shows that the family $(\epsilon^{2\alpha}\mathbf{Q}_\epsilon)^{-1}$ of operators on $L^2(I)$, complemented by the zero operator outside I, approaches an isospectral family in the norm topology.

A similar effect, in a more complicated problem of the behavior of the essential spectra of certain operator families, was studied by Last and Simon in [7].

We will show now that theorems 1.2 and 1.3 imply theorem 1.1. Indeed, the non-zero eigenvalues of the operator $\mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0}$ are the same as those of $\mathbf{Q}_\epsilon^{-1}$. By theorem 1.2 we have for all $j \in \mathbb{N}$ and $\epsilon < \epsilon_0$:

$$\left| \left(l_j(\epsilon) - \frac{\pi^2}{M^2\epsilon^2} \right)^{-1} - l_j^{-1}(\mathbf{Q}_\epsilon) \right| \le \left\| \left(\Delta_\epsilon - \frac{\pi^2}{M^2\epsilon^2}\mathbf{I}_\epsilon \right)^{-1} - \mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0} \right\| \le R_0 \epsilon^{3\alpha};$$

therefore

$$\left| \frac{1}{\epsilon^{2\alpha}\left(l_j(\epsilon) - \frac{\pi^2}{M^2\epsilon^2} \right)} - \frac{1}{\epsilon^{2\alpha} l_j(\mathbf{Q}_\epsilon)} \right| \le R_0 \epsilon^{\alpha}.$$

Further, $l_j(\widehat{\mathbf{Q}}_\epsilon) = \epsilon^{2\alpha} l_j(\mathbf{Q}_\epsilon)$, so that theorem 1.3 implies

$$\epsilon^{2\alpha} l_j(\mathbf{Q}_\epsilon) \to \mu_j$$

which coincides with (1.2).

Theorems 1.2 and 1.3 also allow one to make some conclusions about the behavior of the eigenfunction of the operator Δ_ϵ as $\epsilon \to 0$.

Let $\Psi_{j,\epsilon}(x,y)$ and $\widetilde{\Psi}_{j,\epsilon}(x)$ be the j-s normalized eigenfunctions of the operators Δ_ϵ and $\mathbf{Q}_\epsilon$ respectively. Then theorem 1.2 implies that if the signs of both eigenfunctions are chosen appropriately, then

$$\int_I \left| \left(\mathbf{P}_\epsilon \Psi_{j,\epsilon}\right)(x) - \widetilde{\Psi}_{j,\epsilon}(x) \right|^2 dx \le C_j^2 \epsilon^{6\alpha}. \tag{1.10}$$

Here $\mathbf{P}_\epsilon$ is the orthogonal projection in $L^2(\Omega_\epsilon)$ onto the subspace $\mathcal{L}_\epsilon$. See (3.2) for the explicit formula for $\mathbf{P}_\epsilon$.

Similarly, theorem 1.3 yields

$$\int_I \left| \widetilde{\Psi}_{j,\epsilon}(x) - \epsilon^{-\alpha/2} X_j(x\epsilon^{-\alpha}) \right|^2 dx \to 0. \tag{1.11}$$

where X_j is the j-s normalized eigenfunction of the operator $\mathbf{H}$.

Grieser and Jerison proved in [4] much stronger an estimate for the first eigenfunction in a convex, narrow domain (in our setting, the function $h(x)$ is concave.) Similar problems are discussed in a survey paper [8] by Nazarov.

In the next three sections we prove theorems 1.2 and 1.3. In section 5 we explain the derivation of the inequalities (1.10) and (1.11), and in the last section 6 we describe possible extensions of our main results.

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2. Upper bound for $\|\mathbf{Q}_\epsilon^{-1}\|$

As the first step, we find an upper bound for the quantity $\|\mathbf{Q}_\epsilon^{-1}\|$ as $\epsilon \to 0$. Notice that theorem 1.3 implies $\|\mathbf{Q}_\epsilon^{-1}\| \sim \mu_1^{-1}\epsilon^{2\alpha}$, which is stronger a result than the following lemma.

Lemma 2.1. *Let $h(x)$ meet the properties (i), (ii) of section 1. Then there exists a number $R_1 > 0$ such that*

$$\|\mathbf{Q}_\epsilon^{-1}\| \le R_1 \epsilon^{2\alpha}, \qquad \forall \epsilon > 0. \tag{2.1}$$

Proof. One has (see (1.5))

$$\frac{\epsilon^2 W_\epsilon(x)}{|x|^m} \ge \frac{\pi^2}{|x|^m}\left(\frac{1}{h^2(x)} - \frac{1}{M^2}\right).$$

The function on the right in the last inequality is strictly positive on I and continuous on $[-a, 0]$ and on $[0, b]$ (it equals $2\pi^2 c_\pm M^{-3}$ at $x = 0\pm$; M and $c_\pm$ are numbers from (1.1)). Hence, there exists $\sigma > 0$ such that

$$W_\epsilon(x) \ge \sigma \epsilon^{-2}|x|^m, \qquad \forall \epsilon > 0,\ x \in I. \tag{2.2}$$

The operator $-\frac{d^2}{dx^2} + \sigma|x|^m$ on $L^2(\mathbb{R})$ is positive definite; so

$$\int_{\mathbb{R}} (\chi'(x)^2 + \sigma|x|^m\chi^2(x))dx \ge R_1^{-1}\int_{\mathbb{R}} \chi^2(x)dx, \qquad \forall \chi \in H^1(\mathbb{R})$$

for some positive number R_1. By scaling $x \mapsto \epsilon^{-\alpha}x$ we get

$$\int_{\mathbb{R}} (\chi'(x)^2 + \sigma\epsilon^{-2}|x|^m\chi^2(x))dx \ge R_1^{-1}\epsilon^{-2\alpha}\int_{\mathbb{R}} \chi^2(x)dx, \qquad \forall \chi \in H^1(\mathbb{R}).$$

In particular, this inequality is satisfied for any function $\chi \in H^{1,0}(I)$, extended to the whole of $\mathbb{R}$ by zero. It follows from here and (2.2) that

$$\mathbf{q}_\epsilon[\chi] \ge R_1^{-1}\epsilon^{-2\alpha}\int_I \chi^2 dx, \qquad \forall \chi \in H^{1,0}(I), \tag{2.3}$$

which implies (2.1). □

3. Proof of Theorem 1.2

We will systematically use the orthogonal decomposition

$$L^2(\Omega_\epsilon) = \mathfrak{L}_\epsilon \oplus \mathfrak{L}_\epsilon^\perp, \tag{3.1}$$

and write, for $\psi \in L^2(\Omega_\epsilon)$,

$$\psi = \psi_\chi + U, \qquad \psi_\chi = \mathbf{P}_\epsilon\psi,\ U \perp \mathfrak{L}_\epsilon.$$

Here $\mathbf{P}_\epsilon$ stands for the orthogonal projection in $L^2(\Omega_\epsilon)$ onto the subspace $\mathfrak{L}_\epsilon$. This projection is given by

$$\mathbf{P}_\epsilon\psi = \psi_\chi, \text{ where } \chi(x) = \sqrt{\frac{2}{\epsilon h(x)}}\int_0^{\epsilon h(x)} \psi(x, y)\sin\frac{\pi y}{\epsilon h(x)}dy. \tag{3.2}$$

Note that

$$\mathbf{P}_\epsilon H^{1,0}(\Omega_\epsilon) = H^{1,0}(I).$$

The inclusion $U \in \mathfrak{L}_\epsilon^\perp$ means that

$$\int_0^{\epsilon h(x)} U(x,y)\sin\frac{\pi y}{\epsilon h(x)}dy = 0, \qquad \text{for a.a. } x \in I. \tag{3.3}$$

If $U \in H^{1,0}(\Omega_\epsilon)$, then integration by parts gives

$$\int_0^{\epsilon h(x)} U'_y(x,y)\cos\frac{\pi y}{\epsilon h(x)}dy = 0, \qquad \text{for a.a. } x \in I. \tag{3.4}$$

Because U satisfies the Dirichlet boundary condition, (3.3) implies

$$\begin{aligned}\int_0^{\epsilon h(x)} U^2(x,y)dy &\le \frac{\epsilon^2 h^2(x)}{4\pi^2}\int_0^{\epsilon h(x)} U'_y(x,y)^2 dy\\ &\le \frac{M^2\epsilon^2}{4\pi^2}\int_0^{\epsilon h(x)} U'_y(x,y)^2 dy\end{aligned}$$

and therefore, for $U \in \mathfrak{L}_\epsilon^\perp \cap H^{1,0}(\Omega_\epsilon)$

$$\|U\|^2_{L^2(\Omega_\epsilon)} \le \frac{M^2\epsilon^2}{3\pi^2}\int_{\Omega_\epsilon}\left(|\nabla U|^2 - \frac{\pi^2}{M^2\epsilon^2}|U|^2\right)dxdy. \tag{3.5}$$

In addition, if $U \in H^{1,0}(\Omega_\epsilon)$ then one can differentiate (3.3) with respect to x to get

$$\int_0^{\epsilon h(x)} U'_x(x,y)\sin\frac{\pi y}{\epsilon h(x)}dy = \frac{\pi}{\epsilon}\widetilde{h}(x)\int_0^{\epsilon h(x)} yU(x,y)\cos\frac{\pi y}{\epsilon h(x)}dy. \tag{3.6}$$

Here and later, we use the notation

$$\widetilde{h}(x) = \frac{h'(x)}{h^2(x)};$$

this function repeatedly appears in our calculations.

For the proof of theorem 1.2 we compare the quadratic forms of the operator

$$\mathbf{A}_\epsilon = \Delta_\epsilon - \frac{\pi^2}{M^2\epsilon^2}\mathbf{I}_\epsilon$$

appearing in (1.7), and of its diagonal part with respect to the decomposition (3.1), which is

$$\mathbf{B}_\epsilon = \mathbf{Q}_\epsilon \oplus \big((\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{A}_\epsilon \restriction \mathfrak{L}_\epsilon^\perp\big). \tag{3.7}$$

The quadratic form of $\mathbf{B}_\epsilon$ is (again, for $\psi = \psi_\chi + U$)

$$\mathbf{b}_\epsilon[\psi] = \mathbf{q}_\epsilon[\chi] + \int_{\Omega_\epsilon}\left(|\nabla U|^2 - \frac{\pi^2}{M^2\epsilon^2}|U|^2\right)dxdy,$$

where $\mathbf{q}_\epsilon$ is given by (1.4). From (2.3) and (3.5) we conclude that with some $C > 0$

$$\mathbf{b}_\epsilon[\psi] \geq C\epsilon^{-2\alpha}\|\psi\|^2, \qquad \forall\psi \in H^{1,0}(\Omega_\epsilon). \tag{3.8}$$

The quadratic form of $\mathbf{A}_\epsilon$ is

$$\mathbf{a}_\epsilon[\psi] = \mathbf{b}_\epsilon[\psi] + 2\mathbf{m}_\epsilon[\psi]$$

where (one half of) the off-diagonal term is

$$\mathbf{m}_\epsilon[\psi] = \int_{\Omega_\epsilon} \left(\nabla\psi_\chi \cdot \nabla U - \frac{\pi^2}{M^2\epsilon^2}\psi_\chi U\right) dxdy = \int_{\Omega_\epsilon} (\psi_\chi)'_x U'_x dxdy;$$

the integral containing $(\psi_\chi)'_y U'_y$ vanishes because of (3.4) and the one containing $\psi_\chi U$ vanishes because ψ_χ and U belong to orthogonal subspaces of $L^2(\Omega_\epsilon)$.

We will now estimate $\mathbf{m}_\epsilon[\psi]$. It is convenient to work with the function

$$\phi(x) = h^{-1/2}(x)\chi(x).$$

instead of χ. It is easy to see that for each $\chi \in H^1(I)$ we have

$$\|\phi\| \asymp \|\chi\|, \qquad \|\phi'\|^2 + \|\phi\|^2 \asymp \|\chi'\|^2 + \|\chi\|^2 \leq C\mathbf{q}_\epsilon[\chi]; \tag{3.9}$$

the symbol $\asymp$ stands for two-sided inequality. Taking (3.6) into account, we find that

$$\mathbf{m}_\epsilon[\psi] = \frac{\sqrt{2}\pi}{\epsilon^{3/2}} \int_{\Omega_\epsilon} \widetilde{h}(x) \cos\frac{\pi y}{\epsilon h(x)} \left(\phi' U - \phi U'_x\right) y dxdy. \tag{3.10}$$

The next estimate follows immediately:

$$\begin{aligned} &\mathbf{m}_\epsilon^2[\psi] \\ &\leq C\epsilon^{-3}\left(\|U\|^2_{L^2(\Omega_\epsilon)} \int_{\Omega_\epsilon} \phi'(x)^2 y^3 dxdy + \|U'_x\|^2_{L^2(\Omega_\epsilon)} \int_{\Omega_\epsilon} \phi^2(x) y^3 dxdy\right) \\ &\leq C\left(\|U\|^2_{L^2(\Omega_\epsilon)}\|\phi'\|^2_{L^2(I)} + \|U'_x\|^2_{L^2(\Omega_\epsilon)}\|\phi\|^2_{L^2(I)}\right). \end{aligned}$$

Now we conclude from (3.5), (3.9), and (2.3) that

$$\mathbf{m}_\epsilon^2[\psi] \leq C\left(\epsilon^2\mathbf{b}_\epsilon[U]\mathbf{q}_\epsilon[\chi] + \epsilon^{2\alpha}\mathbf{b}_\epsilon[U]\mathbf{q}_\epsilon[\chi]\right),$$

whence

$$|\mathbf{m}_\epsilon[\psi]| \leq C'\epsilon^\alpha\mathbf{b}_\epsilon[\psi]. \tag{3.11}$$

It is important that the constant C' does not depend on ϵ.

The quadratic form $\mathbf{b}_\epsilon$ is positive definite. Choosing $\epsilon_0 = (4C')^{-1/\alpha}$, we conclude from (3.11) that

$$(1 - C'\epsilon^\alpha)\mathbf{b}_\epsilon[\psi] \leq \mathbf{a}_\epsilon[\psi] \leq (1 + C'\epsilon^\alpha)\mathbf{b}_\epsilon[\psi], \qquad \forall\epsilon < \epsilon_0$$

for all $\psi \in H^1(\Omega_\epsilon)$. Hence, for $\epsilon < \epsilon_0$ the quadratic form $\mathbf{a}_\epsilon$ is also positive definite. Taking (3.8) into account, we find that there exists a positive constant C such that

$$\mathbf{a}_\epsilon[\psi] \geq C^{-1}\epsilon^{-2\alpha}\|\psi\|^2,\ \mathbf{b}_\epsilon[\psi] \geq C^{-1}\epsilon^{-2\alpha}\|\psi\|^2, \qquad \forall \epsilon < \epsilon_0,$$

or, equivalently,

$$\|\mathbf{A}_\epsilon^{-1}\| \leq C\epsilon^{2\alpha},\ \|\mathbf{B}_\epsilon^{-1}\| \leq C\epsilon^{2\alpha}, \qquad \forall \epsilon < \epsilon_0. \tag{3.12}$$

The estimate (3.11) implies an estimate for the bilinear form $\mathbf{m}_\epsilon[\psi_1, \psi_2]$ which corresponds to the quadratic form $\mathbf{m}_\epsilon[\psi]$, i.e.

$$\mathbf{m}_\epsilon[\psi_1, \psi_2] = \int_{\Omega_e} (\psi_{\chi_1})'_x (U_2)'_x dxdy.$$

Namely,

$$|\mathbf{m}_\epsilon[\psi_1, \psi_2]| \leq C'\epsilon^\alpha \left(\mathbf{b}_\epsilon[\psi_1]\mathbf{b}_\epsilon[\psi_2]\right)^{1/2}, \tag{3.13}$$

with the same constant C' as in (3.11).

Note that in the right-hand side of (3.13) each factor $\mathbf{b}_\epsilon[\psi_j]$ can be replaced by $\mathbf{a}_\epsilon[\psi_j]$; that will result in a change of the constant C', which is not essential.

We have, for any $\psi_1, \psi_2 \in H^{1,0}(\Omega_\epsilon)$:

$$\begin{aligned} \left|(\mathbf{A}_\epsilon^{1/2}\psi_1, \mathbf{A}_\epsilon^{1/2}\psi_2) - (\mathbf{B}_\epsilon^{1/2}\psi_1, \mathbf{B}_\epsilon^{1/2}\psi_2)\right| &= |\mathbf{a}_\epsilon[\psi_1, \psi_2] - \mathbf{b}_\epsilon[\psi_1, \psi_2]| \\ &= 2|\mathbf{m}_\epsilon[\psi_1, \psi_2]| \leq C\epsilon^\alpha(\mathbf{a}_\epsilon[\psi_1]\mathbf{b}_\epsilon[\psi_2])^{1/2}. \end{aligned}$$

Take here $\psi_1 = \mathbf{B}_\epsilon^{-1}f,\ \psi_2 = \mathbf{A}_\epsilon^{-1}g$, where $f, g \in L^2(\Omega_\epsilon)$ are arbitrary. Then we get by (3.12):

$$|(\mathbf{A}_\epsilon^{-1}f, g) - (\mathbf{B}_\epsilon^{-1}f, g)| \leq C\epsilon^\alpha\big((\mathbf{A}_\epsilon^{-1}g, g)(\mathbf{B}_\epsilon^{-1}f, f)\big)^{1/2} \leq C\epsilon^{3\alpha}\|f\|\|g\|;$$

therefore

$$\|\mathbf{A}_\epsilon^{-1} - \mathbf{B}_\epsilon^{-1}\| \leq C\epsilon^{3\alpha}. \tag{3.14}$$

It follows from (3.7) and (3.5) that

$$\|\mathbf{B}_\epsilon^{-1} - (\mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0})\| = \|\big((\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{A}_\epsilon \restriction \mathcal{L}_\epsilon^\perp\big)^{-1}\| \leq M^2\epsilon^2/3\pi^2.$$

Together with (3.14), this completes the proof of theorem 1.2.

4. Proof of theorem 1.3

In the course of the proof we rely upon the following statement.

Proposition 4.1. 1° *Let $V(x) \geq 0$ be a measurable function on $\mathbb{R}$, such that $V(x) \to \infty$ as $|x| \to \infty$, and let $\{I_\epsilon\}$, $0 < \epsilon < 1$, be an expanding family of intervals:*

$$I_{\epsilon_1} \subset I_{\epsilon_2} \ (\epsilon_1 > \epsilon_2), \qquad \cup_{0<\epsilon<1} I_\epsilon = \mathbb{R}.$$

Consider the quadratic form

$$\mathbf{z}_V[u] = \int_{\mathbb{R}} (u'^2 + Vu^2)dx, \qquad u \in \mathfrak{d}_V := \{H^1(\mathbb{R}) : \mathbf{z}_V[u] < \infty\}$$

and a family of its restriction $\mathbf{z}_{V,I_\epsilon}$ to the domains

$$\mathfrak{d}_{V,I_\epsilon} = \{u \in \mathfrak{d}_V : u\,|_{\partial I_\epsilon} = 0\}.$$

Let $\mathbf{Z}_V, \mathbf{Z}_{V,I_\epsilon}$ stand for the corresponding self-adjoint operators on $L^2(\mathbb{R})$. Then

$$\|\mathbf{Z}_{V,I_\epsilon}^{-1} - \mathbf{Z}_V^{-1}\| \to 0, \qquad \epsilon \to 0, \tag{4.1}$$

2° *Let a potential $V_0 \geq 0$ be fixed, such that $V_0(x) \to \infty$ as $|x| \to \infty$. Then the convergence in (4.1) is uniform in the class of all potentials V such that*

$$V(x) \geq V_0(x) \qquad \text{on } \mathbb{R}.$$

Proof. 1° Under the assumptions of proposition the strong convergence $\mathbf{Z}_{V,I_\epsilon}^{-1} \to \mathbf{Z}_V^{-1}$ is well known. For instance, it follows from theorem VIII.1.5 in the book [5]. Its assumptions are evidently satisfied if we take into account that $C_0^\infty(\mathbb{R})$ is a core for the operator $\mathbf{Z}_V$.

For each ϵ we have $\mathfrak{d}_{V,I_\epsilon} \subset \mathfrak{d}_V$ and $\mathbf{z}_{V,I_\epsilon}[u] = \mathbf{z}_V[u]$ for every $u \in \mathfrak{d}_{V,\epsilon}$. By the definition of inequalities between self-adjoit operators (see e.g. [1], section 10.2.3), this means that $\mathbf{Z}_{V,I_\epsilon} \geq \mathbf{Z}_V$ and, by theorem 10.2.6 from [1],

$$\mathbf{Z}_{V,I_\epsilon}^{-1} \leq \mathbf{Z}_V^{-1}.$$

Since $V(x) \to \infty$ as $|x| \to \infty$, the operator $\mathbf{Z}_V^{-1}$ is compact.

Now, we get the statement 1° by applying theorem 2.16 in [11] (which is an analogue of the classical Lebesgue theorem on dominated convergence). In particular, the theorem says that if $\mathbf{T}_\epsilon$, $0 < \epsilon < \epsilon_0$ is a family of compact, self-adjoint operators such that $\mathbf{T}_\epsilon \to \mathbf{T}$ strongly, and there exists a compact non-negative operator $\mathbf{T}_0$, such that $|\mathbf{T}_\epsilon| \leq \mathbf{T}_0$ for each ϵ, then $\|\mathbf{T}_\epsilon - \mathbf{T}\| \to 0$.

2° Actually, this statement also is a consequence of theorem 2.16 in [11]. More exactly, it immediately follows from the last displayed inequality in the proof of theorem. $\square$

Each operator $\mathbf{Z}_{V,I_\epsilon}$ appearing in the formulation is the direct sum of the operators inside and outside the interval I_ϵ, generated by the differential expression $-d^2/dx^2 + V$ and the Dirichlet conditions at ∂I_ϵ. Let us denote these operators as $\mathbf{Z}_{V,int(I_\epsilon)},\ \mathbf{Z}_{V,ext(I_\epsilon)}$ respectively.

Corollary 4.2. *Both statements of proposition 4.1 remain valid if we replace each operator* $\mathbf{Z}^{-1}_{V,I_\epsilon}$ *by* $\mathbf{Z}^{-1}_{V,int(I_\epsilon)} \oplus \mathbf{0}$*, where* $\mathbf{0}$ *is the zero operator on the subspace* $\{u \in L^2(\mathbb{R}) : u = 0 \text{ on } \mathbb{R} \setminus I_\epsilon\}$.

Indeed, this immediately follows from the fact that

$$(\mathbf{Z}_{V,ext(I_\epsilon)}u, u) \geq \|u\|^2 \inf_{x \in \mathbb{R}\setminus I_\epsilon} V(x),$$

whence $\|\mathbf{Z}^{-1}_{V,ext(I_\epsilon)}\| \to 0$.

Proof of theorem 1.3. Let W_ϵ be the function defined in (1.5) and

$$V_\epsilon(t) = \epsilon^{2\alpha} W_\epsilon(t\epsilon^\alpha).$$

Then the quadratic form of the operator (1.8) is

$$\widehat{\mathbf{q}}_\epsilon[u] = \int_{I_\epsilon} \big(u'(t)^2 + V_\epsilon(t)u^2(t)\big)\,dt.$$

The assumption (1.1), say for $x > 0$, can be written as

$$h(x) = M - c_+x^m + \rho(x)x^{m+1}, \qquad \rho \in L^\infty(0, b).$$

Hence,

$$\frac{1}{h^2(x)} - \frac{1}{M^2} = 2c_+M^{-3}x^m + \rho_1(x)x^{m+1}, \qquad \rho_1 \in L^\infty(0, b);$$

therefore

$$V_\epsilon(t) = q(t) + \pi^2\rho_1(t\epsilon^\alpha)t^{m+1}\epsilon^\alpha + \epsilon^{2\alpha}v(t\epsilon^\alpha), \qquad t \in (0, b\epsilon^{-\alpha}).$$

A similar equality is satisfied also for $t \in (-a\epsilon^{-\alpha}, 0)$.

Along with $\{I_\epsilon\}$, we need a system $\{I'_\epsilon\}$ of narrower intervals, say

$$I'_\epsilon = (-\epsilon^{-\beta}, \epsilon^{-\beta}).$$

Here $\beta > 0$ can be taken arbitrary; the only condition is $\beta(m+1) < \alpha$. Then for any $\eta > 0$ there exists a number $\epsilon(\eta) > 0$, such that

$$|V_\epsilon(t) - q(t)| < \eta \qquad \text{for all } t \in I'_\epsilon,\ \epsilon < \epsilon(\eta). \tag{4.2}$$

In addition, it follows from (2.2) that

$$V_\epsilon(t) \geq \sigma|t|^m \qquad \text{for all } t \in I_\epsilon,\ \epsilon < 1. \tag{4.3}$$

It is useful to extend each function $V_\epsilon(t)$ to the whole of $\mathbb{R}$, taking $V_\epsilon(t) = \sigma|t|^m$ for $t \notin I_\epsilon$. With each (extended) function V_ϵ we associate three operators: $\mathbf{H}_\epsilon$ acting on $L^2(\mathbb{R})$, $\widehat{\mathbf{Q}}_\epsilon$ acting on $L^2(I_\epsilon)$, and $\widehat{\mathbf{Q}'}_\epsilon$

acting on $L^2(I'_\epsilon)$. Each operator acts as $-d^2/dt^2 + V_\epsilon(t)$; the last two operators are taken with the Dirichlet boundary conditions. To apply proposition 4.1, one takes

$$\mathbf{H}_\epsilon = \mathbf{Z}_{V_\epsilon}; \qquad \widehat{\mathbf{Q}}_\epsilon = \mathbf{Z}_{V_\epsilon, int(I_\epsilon)},\ \widehat{\mathbf{Q}'}_\epsilon = \mathbf{Z}_{V_\epsilon, int(I'_\epsilon)}.$$

In particular, $\widehat{\mathbf{Q}}_\epsilon$ is nothing but the operator (1.8).

By proposition 4.1, 2° we have

$$\|\widehat{\mathbf{Q}}_\epsilon^{-1} \oplus \mathbf{0} - \mathbf{H}_\epsilon^{-1}\| \to 0, \qquad \|\widehat{\mathbf{Q}'}_\epsilon^{-1} \oplus \mathbf{0} - \mathbf{H}_\epsilon{}^{-1}\| \to 0$$

as $\epsilon \to 0$. Here $\mathbf{0}$ stands for the zero operator on $L^2(\mathbb{R}) \ominus L^2(I_\epsilon)$, or on $L^2(\mathbb{R}) \ominus L^2(I'_\epsilon)$. Therefore,

$$\|\widehat{\mathbf{Q}}_\epsilon^{-1} \oplus \mathbf{0} - \widehat{\mathbf{Q}'}_\epsilon^{-1} \oplus \mathbf{0}\| \to 0. \tag{4.4}$$

Note that the operators $\widehat{\mathbf{Q}}_\epsilon$, $\widehat{\mathbf{Q}'}_\epsilon$ depend on the parameter ϵ in two ways: via the potential and via the interval. For this reason, the statement 1° of proposition 4.1 is insufficient for making these conclusions.

Consider also the family of operators $\mathbf{H}'_\epsilon := \mathbf{Z}_{q(t), int(I'_\epsilon)}$. They act on $L^2(I'_\epsilon)$ as

$$\mathbf{H}'_\epsilon u = -u'' + q(t)u,$$

with the Dirichlet conditions at $\partial I'_\epsilon$. This time, the potential does not involve the parameter ϵ, and we conclude from proposition 4.1, 1° that

$$\|\mathbf{H}'^{-1}_\epsilon \oplus \mathbf{0} - \mathbf{H}^{-1}\| \to 0. \tag{4.5}$$

In addition, one has

$$\|\widehat{\mathbf{Q}'}_\epsilon^{-1} - \mathbf{H}'^{-1}_\epsilon\| \to 0, \qquad \epsilon \to 0. \tag{4.6}$$

Indeed, by Hilbert's resolvent formula,

$$\widehat{\mathbf{Q}'}_\epsilon^{-1} - \mathbf{H}'^{-1}_\epsilon = -\widehat{\mathbf{Q}'}_\epsilon^{-1}(V_\epsilon(t) - q(t))\mathbf{H}'^{-1}_\epsilon.$$

Here $\|\widehat{\mathbf{Q}'}_\epsilon^{-1}\|$, $\|\widehat{\mathbf{H}'}_\epsilon^{-1}\| \le C$ for all $\epsilon < 1$ (this follows from (4.3)), and by (4.2) the norm of the multiplication operator is smaller than an arbitrary η, provided that ϵ is small.

Theorem 1.3 (that is, eq. (1.9)) immediately follows from (4.4), (4.5), and (4.6).

5. Eigenfunction convergence

5.1. We start from some elementary remarks from the Hilbert space theory. Let e, f be normalized elements of a Hilbert space $\mathcal{H}$, and

$$\mathbf{K} = (\cdot, e)e - (\cdot, f)f. \tag{5.1}$$

A direct calculation shows that the Hilbert-Schmidt norm of the operator $\mathbf{K}$ is given by

$$\|\mathbf{K}\|_{HS}^2 = 2(1 - |(e, f)|^2).$$

Suppose now that (e, f) is real, then $\|e \pm f\|^2 = 2(1 \pm (e, f))$ and hence,

$$\min(\|e - f\|, \|e + f\|) \leq (\|e - f\| \|e + f\|)^{1/2} = \sqrt{2}\, \|\mathbf{K}\|_{HS}. \tag{5.2}$$

Further, let $\mathcal{H}$ be decomposed into an orthogonal sum of two subspaces,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

and let $f \in \mathcal{H}_0$. Then, along with (5.2), one has

$$\min(\|\mathbf{P}_0 e - f\|, \|\mathbf{P}_0 e + f\|) \leq \sqrt{2}\, \|\mathbf{K}\|_{HS} \tag{5.3}$$

where $\mathbf{P}_0$ is the operator of orthogonal projection onto $\mathcal{H}_0$.

Suppose now that $\mathbf{S}, \mathbf{T}$ are two self-adjoint operators in $\mathcal{H}$. We assume that they are bounded, though this is actually not needed. Suppose that, on some interval $\delta \in \mathbb{R}$, each operator has exactly one point of spectrum, and this point is a simple eigenvalue. Say, l_0, μ_0 are these eigenvalues for $\mathbf{S}, \mathbf{T}$ respectively, and e, f are the corresponding normalized eigenvectors. Let $\phi(l)$ be a smooth, real-valued function on $\mathbb{R}$, which vanishes outside δ and is such that

$$\phi(l_0) = \phi(\mu_0) = 1.$$

Then we conclude from Spectral Theorem that

$$\phi(\mathbf{S}) = (\cdot, e)e; \qquad \phi(\mathbf{T}) = (\cdot, f)f,$$

and therefore the operator $\mathbf{K}$ can be represented as

$$\mathbf{K} = \phi(\mathbf{S}) - \phi(\mathbf{T}).$$

This representation allows us to apply the theory of double operator integrals (see [2], and especially section 8 therein.) In particular, we conclude from theorems 8.1 and 8.3 that

$$\|\mathbf{K}\| \leq C \|\mathbf{S} - \mathbf{T}\|, \qquad C = C(\phi).$$

Since $\operatorname{rank} \mathbf{K} \leq 2$, we conclude that also

$$\|\mathbf{K}\|_{HS} \leq \sqrt{2}\, \|\mathbf{K}\| \leq C\sqrt{2}\, \|\mathbf{S} - \mathbf{T}\|. \tag{5.4}$$

5.2. Now we proceed to proving (1.10) and (1.11). We start from (1.11). Then we take $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{H}_0 = L^2(I_\epsilon)$. Recall that we identify $L^2(I_\epsilon)$ with the subspace in $L^2(\mathbb{R})$ formed by functions vanishing outside I_ϵ. The operator $\mathbf{P}_0$ acts as the restriction operator from $\mathbb{R}$ to the interval I_ϵ.

We apply the estimate (5.4) to the operators $\mathbf{S} = \mathbf{H}^{-1}$ and $\mathbf{T} = \widehat{\mathbf{Q}}_\epsilon^{-1} \oplus \mathbf{0}$. Theorem 1.3 guarantees that for each $j \in \mathbb{N}$ there exists a number ϵ_j^*, such that for any $\epsilon < \epsilon_j^*$ there is a neighborhood δ of the point μ_j^{-1}, in which the spectrum of $\widehat{\mathbf{Q}}_\epsilon^{-1}$ reduces to a single and simple eigenvalue. By (1.8), this eigenvalue is $\epsilon^{-2\alpha} l_j^{-1}(\mathbf{Q}_\epsilon)$.

The eigenfunction of $\mathbf{S}$ which corresponds to the eigenvalue μ_j^{-1} is $X_j(t)$, and that of $\mathbf{T}$ which corresponds to the eigenvalue $\epsilon^{-2\alpha} l_j^{-1}(\mathbf{Q}_\epsilon)$ is equal to $\epsilon^{\alpha/2}\widetilde{\Psi}_{j,\epsilon}(t\epsilon^\alpha)$ on I and vanishes outside I. Under the appropriate choice of the sign of $\widetilde{\Psi}_{j,\epsilon}(x)$, we conclude from (5.4) that

$$\int_{I_\epsilon} \left| X_j(t) - \epsilon^{\alpha/2}\widetilde{\Psi}_{j,\epsilon}(t\epsilon^\alpha) \right|^2 dt \to 0.$$

Using the substitution $t = x\epsilon^{-\alpha}$, we get (1.11).

To get (1.10), we apply the estimate (5.4) to the operators

$$\mathbf{S} = \mathbf{A}_\epsilon^{-1} = (\Delta_\epsilon - \frac{\pi^2}{M^2\epsilon^2}\mathbf{I}_\epsilon)^{-1}, \qquad \mathbf{T} = \mathbf{Q}_\epsilon^{-1} \oplus \mathbf{0}.$$

Here $\mathcal{H} = L^2(\Omega_\epsilon)$, $\mathcal{H}_0 = \mathcal{L}_\epsilon$, and $\mathbf{P}_0$ is the operator $\mathbf{P}_\epsilon$ described in (3.2). Eigenvalues of $\mathbf{S}$ are $l_j(\mathbf{S}) = (l_j(\epsilon) - \frac{\pi^2}{M^2\epsilon^2})^{-1}$, and the corresponding eigenfunctions are $\Psi_{j,\epsilon}(x,y)$. Recall that $l_j(\epsilon)$ is our notation for $l_j(\Delta_\epsilon)$. By theorem 1.2, for each $j \in \mathbb{N}$ there exists an $\epsilon_j^* > 0$, such that for $\epsilon < \epsilon_j^*$ the point $l_j(\mathbf{S})$ has a neighborhood $\delta \subset \mathbb{R}$ containing exactly one eigenvalue of the operator $\mathbf{T}$. This eigenvalue is simple and necesarily coincides with $l_j(\mathbf{Q}_\epsilon)$. The corresponding eigenfunction is $\psi_\chi(x,y)$ with $\chi(x) = \widetilde{\Psi}_{j,\epsilon}(x)$. Now, the inequality (5.4) turns into (1.10).

Note that the interval δ appearing in this argument is quite narrow for large values of j. Indeed, its length can not exceed the number

$$\left(l_{j-1}(\epsilon) - \frac{\pi^2}{M^2\epsilon^2}\right)^{-1} - \left(l_{j+1}(\epsilon) - \frac{\pi^2}{M^2\epsilon^2}\right)^{-1}.$$

This results in a very large constant $C = C_j$ in the corresponding inequality (1.10).

6. Remarks on possible extensions

1. The result of theorem 1.1 extends to the case when $h(x)$ is continuous on I, positive inside I, satisfies the condition (i), and in a neighborhood of $x = 0$ admits the expansion (1.1); however, the function $h(x)$ is allowed to vanish at endpoints of I. A simple but important example of such a function is $h(t) = 1 - |x|$ on the segment $I = [-1, 1]$.

This statement is easy to justify by using the variatonal principle in its simplest form. Namely, we construct functions $h^{\pm}$ in such a way that h^+ satisfies the conditions (i) and (ii) on I, with the same coefficients in the expansion (1.1), and the inequality $h(x) \le h^+(x)$, while h^- satisfies the conditions (i) and (ii) on a smaller segment $\widetilde{I} \subset I$, also with the same coefficients in (1.1), and the inequality $h(x) \ge h^-(x)$, $x \in \widetilde{I}$. Denote

$$\Omega_\epsilon^+ = \{(x, y) : x \in I,\ 0 < y < \epsilon h^+(x)\};$$
$$\Omega_\epsilon^- = \{(x, y) : x \in \widetilde{I},\ 0 < y < \epsilon h^-(x)\}.$$

Let $l_j^{\pm}(\epsilon)$ stand for the eigenvalues of the Dirichlet Laplacian in $\Omega_\epsilon^{\pm}$, then by the variatonal principle we have

$$l_j^+(\epsilon) \le l_j(\epsilon) \le l_j^-(\epsilon).$$

By theorem 1.1, the equality (1.2) is valid for $l_j^{\pm}(\epsilon)$; therefore it holds also for $l_j(\epsilon)$.

It remains to construct the functions $h^{\pm}(x)$. Let $x > 0$. It follows from (1.1) that on some segment $[0, \eta]$ we have

$$|M - h(x) - c_+ x^m| \le K x^{m+1},$$

with some constant $K > 0$. The function $h^+(x)$ can be obtained (for $x > 0$) as an appropriate extension of $M - c_+ x^m + K x^{m+1}$ to $[0, b]$, and $h^-(x)$ can be obtained as the restriction of $M - c_+ x^m - K x^{m+1}$ to a segment $[0, \widetilde{\eta}]$, where $\widetilde{\eta} \le \eta$ is small enough, to guarantee $h^-(x) > 0$ on $[0, \widetilde{\eta}]$. For $x < 0$, we construct $h^{\pm}(x)$ in a similar way.

At the moment it is unclear to the authors, whether theorems 1.2 and 1.3 also extend to the case when $h(x)$ vanishes at the endpoints of the interval I. The main technical obstacle comes from the fact that the function $\sin \frac{\pi y}{\epsilon h(x)}$ oscillates very fast near the points where $h(x)$ vanishes.

2. The results of all three theorems 1.1 – 1.3 extend to the case when the Dirichlet conditions at $x = -a, x = b$ are replaced by the Neumann conditions. The argument is basically the same as for the Dirichlet

problem, except for two important points: the proofs of lemma 2.1 and of proposition 4.1 do not apply to the Neumann case.

These obstacles can be overcome, so that the results survive under the same assumptions (i), (ii) on the function $h(x)$.

3. Theorems 1.2 and 1.3 extend to the case of an infinite strip, when I is the whole line, or a half-line. Let, say, $I = \mathbb{R}$. One has to impose additional conditions on the behavior of $h(x)$ as $|x| \to \infty$. A simple condition is

(iii) The function $h(x)$ is such that

$$\limsup_{|x|\to\infty} h(x) < M; \qquad \frac{h'}{h} \in L^\infty(\mathbb{R}).$$

Theorem 6.1 below, which is an analogue of theorem 1.1, looks a little bit more complicated than the latter, since the spectrum of the Dirichlet Laplacian Δ_ϵ in Ω_ϵ is now not necessarily discrete. In the formulation of the theorem $\nu(\epsilon)$ stands for the bottom of of the essential spectrum of Δ_ϵ, and we take $\nu(\epsilon) = \infty$ if the spectrum of Δ_ϵ is discrete. We denote by $n_-(\epsilon)$, $n_-(\epsilon) \le \infty$, the number of eigenvalues $l_j(\epsilon) < \nu(\epsilon)$.

Theorem 6.1. *If $h(x)$ satisfies the conditions (i), (ii) and (iii), then for ϵ small the spectrum of Δ_ϵ below $\nu(\epsilon)$ is non-empty and $n_-(\epsilon) \to \infty$ as $\epsilon \to 0$. For each $j \in \mathbb{N}$ the equality* (1.2) *holds, where again, μ_j are eigenvalues of the operator* (1.3).

Proof of an analogue of theorem 1.3 turns out to be the crucial step in the analysis of the case $I = \mathbb{R}$. The main difficulty here is that theorem 2.16 in [11] does not apply, since the operators involved may be non-compact. However, an appropriate substitute can be proved, and this leads to the desired result.

A detailed exposition of the material related to remarks 2 and 3 will be given in a forthcoming paper.

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